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RECURRENCE CONDITIONS IN DENUMERABLE STATE MARKOV DECISION PROCESSES

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Recurrence conditions in denumerable state Markov decision processes *)

by

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ABSTRACT

This survey paper considers an undiscounted semi-Markov decision problem with denumerable state space and compact metric action spaces. Recurrence conditions on the transition probability matrices associated with the stationary policies are considered and relations between these conditions are established. Also it is shown that under each of these conditions the optimality equation for the average costs has a bounded solution.

KEY WORDS & PHRASES: *semi-Markov decision processes, denumerable state space, compact metric action spaces, recurrence conditions, equivalences, average costs, optimality equation.*

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*) This report will be submitted for publication elsewhere.

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1. INTRODUCTION

In this paper we consider an undiscounted semi-Markov decision model specified by five objects $(I, A(i), p_{ij}(a), c(i,a), \tau(i,a))$. We are concerned with a dynamic system which at decision epochs beginning with epoch 0 is observed to be in one of the states of the *denumerable* state space I. After observing the state of the system, an action must be chosen. For any state $i \in I$, the set $A(i)$ denotes the set of possible actions for state i. If the system is in state i at any decision epoch and action $a \in A(i)$ is chosen, then regardless of the history of the system, the following happens:

(i) an immediate cost $c(i,a)$ is incurred;

(ii) the time until the next decision epoch is random with mean $\tau(i,a)$;

(iii) at the next decision epoch the system will be in state j with probability $p_{ij}(a)$ where $\Sigma_{j \in I} p_{ij}(a) = 1$ for all $i \in I$ and $a \in A(i)$.

Unless stated otherwise, we make throughout this paper the following assumptions.

A1. *For any* $i \in I$, *the set* $A(i)$ *is a compact metric space on which both* $c(i,a)$, $\tau(i,a)$ *and* $p_{ij}(a)$ *for any* $j \in I$ *are continuous.*

A2. *There is a finite number* M *such that* $|c(i,a)| \le M$ *and* $\tau(i,a) \le M$ *for all* $i \in I$ *and* $a \in A(i)$.

A3. *There is a positive number* δ *such that* $\tau(i,a) \ge \delta$ *for all* $i \in I$ *and* $a \in A(i)$.

We note that Assumption A1 is satisfied when $A(i)$ is finite for all $i \in I$.

A policy π for controlling the system is any (possibly randomized) rule for choosing actions. For any initial state i and policy π, denote by X_n and a_n the state and the action chosen at the nth decision epoch for $n = 0,1,\ldots$. (the 0^{th} decision epoch is at epoch 0). Denote by E_π the expectation when policy π is used. Let $F = \times_{i \in I} A(i)$, i.e. F is the class of all functions f which add to each state $i \in I$ a single action $f(i) \in A(i)$. For any $f \in F$, denote by $f^{(\infty)}$ the stationary policy which prescribes action $f(i)$ whenever the system is in state i. Under each stationary policy $f^{(\infty)}$ the process $\{X_n, n \ge 0\}$ is a Markov-chain with one-step transition probability matrix $P(f) = (p_{ij}(f(i)))$, $i,j \in I$. For $n = 1,2,\ldots$, denote the n-step transition probability matrix of this Markov chain by $P^n(f) = (p^n_{ij}(f))$, $i,j \in I$.

In this survey paper which is based on results in [3] and [4] we shall study a number of recurrence conditions on the stochastic matrices $P(f)$, $f \in F$. In section 2 we give these conditions and prove several relations between them.

We discuss in section 3 the optimality equation for the average costs and verify that under each of the above conditions this optimality equation has a bounded solution.

2. RECURRENCE CONDITIONS

We first introduce the following notation. For any set $A \subseteq I$, define

$$N_A = \min\{n \geq 1 \mid X_n \in A\} \text{ where } N_A = \infty \text{ if } X_n \notin A \text{ for all } n \geq 1.$$

Consider now the following recurrence conditions C1-C5 on the stochastic matrices $P(f)$, $f \in F$.

C1. *There is a finite set* K *and a finite number* B *such that*

(2.1) $\quad E_{f^{(\infty)}}\{N_K \mid X_0 = i\} \leq B$ *for all* $i \in I$ *and* $f \in F$.

Further for any $f \in F$ *the stochastic matrix* $P(f)$ *has no two disjoint closed sets.*

C2. *There is a finite set* K *and a finite number* B *such that for any* $f \in F$ *a state* $s_f \in K$ *exists for which*

(2.2) $\quad E_{f^{(\infty)}}\{N_{\{s_f\}} \mid X_0 = i\} \leq B$ *for all* $i \in I$.

C3. *There is a finite set* K, *an integer* $\nu \geq 1$ *and a number* $\rho > 0$ *such that*

(2.3) $\quad \sum_{j \in K} p_{ij}^{\nu}(f) \geq \rho$ *for all* $i \in I$ *and* $f \in F$.

Further, for any $f \in F$ *the stochastic matrix* $P(f)$ *has no two disjoint closed sets.*

C4. *There is an integer* $\nu \geq 1$ *and a number* $\rho > 0$ *such that*

(2.4) $\quad \sum_{j \in I} \min[p_{i_1 j}^{\nu}(f), p_{i_2 j}^{\nu}(f)] \geq \rho$ *for all* $i_1, i_2 \in I$ *and* $f \in F$.

C5. *There is an integer* $\nu \geq 1$ *and a number* $\rho > 0$ *such that for each* $f \in F$ *a probability distribution* $\{\pi_j(f), j \in I\}$ *(say) exists for which*

(2.5) $\quad \left| \sum_{j \in A} p_{ij}^{n}(f) - \sum_{j \in A} \pi_j(f) \right| \leq (1-\rho)^{[n/\nu]}$ *for all* $A \subseteq I$, $j \in I$ *and* $n \geq 1$,

where [x] *denotes the largest integer less than or equal to* x.

The condition C1 was considered in [4], cf. also [10]. Clearly condition C2 implies C1. The condition C3 was introduced in [4] and called the simultaneous Doeblin condition since for each $f\in F$ the stochastic matrix $P(f)$ satisfies the so-called Doeblin condition from Markov chain theory. The conditions C4 and C5 were introduced in [3]. Following Markov chain terminology, the conditions C4 and C5 could be called a simultaneous scrambling condition (cf. [15]) and a simultaneous quasi-compactness condition (cf. [9]) respectively. Observe that under each of the above conditions any stochastic matrix $P(f)$, $f\in F$ has no two disjoint closed sets. Further, any $P(f)$ is aperiodic under both C4 and C5. Finally, we note that the left side of (2.4) denotes the ergodic coefficient of the stochastic matrix $P^{\nu}(f)$ and that $\{\pi_j(f),\ j\in I\}$ in C5 denotes the unique stationary probability distribution of $P(f)$.

Before proving a number of relations between the above conditions, we first mention the following facts which will be frequently used hereafter. Since $F=\times_{i\in I} A(i)$, we have by A1 that the set F is a *compact metric space* in the product topology. Further, using the relation

$$(2.6)\qquad p_{ij}^{m+1}(f) = \sum_{k\in I} p_{ik}(f)\,p_{kj}^{m}(f) \quad \text{for all } i,j\in I,\ m\geq 1 \text{ and } f\in F.$$

and Proposition 18 on p. 232 in [11], it immediately follows by induction that for any $n\geq 1$ and $i,j\in I$ the function $p_{ij}^{n}(f)$ is continuous on F.*

From Markov chain theory we have that for any $f\in F$

$$(2.7)\qquad \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} p_{ij}^{k}(f) = \pi_{ij}(f) \text{ (say) exists for all } i,j\in I$$

In case $P(f)$ has no two disjoint closed sets, then

$$(2.8)\qquad \pi_{ij}(f) = \pi_j(f) \text{ (say) for all } i,j\in I$$

where the non-negative numbers $\pi_j(f)$ satisfy

$$(2.9)\qquad \pi_j(f) = \sum_{i\in I} p_{ij}(f)\pi_i(f) \text{ for all } j\in I.$$

We note that additional assumptions are needed to ensure that $\{\pi_j(f)\}$ in (2.8) is a probability distribution in which case $\{\pi_j(f), j\in I\}$ is the unique probability distribution satisfying (2.9).

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*In the remainder of this section we shall not use the product property $F=\times A(i)$ but only the fact that F is a compact metric space.

We now first prove

THEOREM 2.1 (cf. [4]). *Suppose for any* $f \in F$ *that the stochastic matrix* $P(f)$ *has no two disjoint closed sets and that* $\{\pi_j(f), j \in I\}$ *is a probability distribution. Then the function* $\pi_j(f)$ *is continuous on* F *for each* $j \in I$ *if and only if for each* $\varepsilon > 0$ *there is a finite set* $K(\varepsilon)$ *such that*

$$(2.10) \qquad \sum_{j \in K(\varepsilon)} \pi_j(f) \geq 1-\varepsilon \text{ for all } f \in F.$$

PROOF. Suppose first that for each $\varepsilon > 0$ we can find a finite set $K(\varepsilon)$ such that (2.10) holds. Now, let $\{f_n, n \geq 1\}$ be any sequence in F such that $f_n \to f^*$ as $n \to \infty$. Choose $h \in I$. We shall now verify that

$$(2.11) \qquad \lim_{n\to\infty} \pi_h(f_n) = \pi_h(f^*).$$

To do this, let α_h be any limit point of $\{\pi_h(f_n), n \geq 1\}$. By the well-known diagonalization method, we can choose a subsequence $\{n_k, k \geq 1\}$ of integers for which

$$\lim_{k\to\infty} \pi_j(f_{n_k}) = \pi_j \text{ (say) exists for all } j \in I \text{ such that } \pi_h = \alpha_h.$$

Take $f = f_{n_k}$ in (2.9) and let $k \to \infty$. Using the fact that $p_{ij}(f)$ is continuous on F for all i,j and using Proposition 18 on p. 232 in [11], we find

$$(2.12) \qquad \pi_j = \sum_{k \in I} p_{kj}(f^*)\pi_k \quad \text{for all } j \in I.$$

Further, using (2.10), we have

$$(2.13) \qquad \sum_{j \in I} \pi_j = 1.$$

By (2.12)-(2.13) and the fact that $P(f^*)$ has a unique stationary probability distribution, it follows that $\pi_j = \pi_j(f^*)$ for all $j \in I$. In particular $\alpha_h = \pi_h(f^*)$, which verifies (2.11).

Suppose next that $\pi_j(f)$ is continuous on F for each $j \in I$. Let now $\{K_n, n \geq 1\}$ be any sequence of finite subsets of I such that

$$K_{n+1} \supseteq K_n \text{ for all } n \geq 1 \text{ and } \lim_{n\to\infty} K_n = I.$$

Define for n=1,2,....,

$$a_n(f) = \sum_{j\in K_n} \pi_j(f), \ f\in F.$$

Then $a_n(f)$ is continuous on F for all $n\geq 1$. Further, we have for any $f\in F$ that

$$a_{n+1}(f)\geq a_n(f) \text{ for all } n\geq 1 \text{ and } \lim_{n\to\infty} a_n(f) = 1.$$

Now, by Theorem 7.13 in [12], it follows that $a_n(f)$ converges to 1 uniformly in $f\in F$ as $n\to\infty$. Hence for each $\varepsilon>0$ we can find a finite n such that $a_n(f)\geq 1-\varepsilon$ which verifies (2.10).

We note that (2.10) states that the collection $[\{\pi_j(f), j\in I\}|f\in F]$ of probability distributions is *tight*.

THEOREM 2.2 (cf. [4]). *The following three conditions are equivalent*
(i) *Condition C3 without the requirement that for any* $f\in F$ *the stochastic matrix* P(f) *has no two disjoint closed sets.*
(ii) *There is a finite set* K *and a finite number* B *such that for all* $i\in I$ *and* $f\in F$

$$\text{(2.14)} \qquad E_{f^{(\infty)}}\{N_K|X_0=i\}\leq B.$$

(iii) *For any* $\varepsilon>0$ *there is a finite set* $K(\varepsilon)$ *and an integer* $\nu(\varepsilon)\geq 1$ *such that*

$$\text{(2.15)} \qquad \sum_{j\in K(\varepsilon)} p_{ij}^{\nu(\varepsilon)}(f)\geq 1-\varepsilon \text{ for all } i\in I \text{ and } f\in F.$$

PROOF. Suppose first that (i) holds with triple (K,ν,ρ) in C3. We shall verify (ii). Now,

$$\Pr_{f^{(\infty)}}\{X_n\notin K \text{ for } 1\leq n\leq \nu|X_0=i\}\leq 1-\rho \text{ for all } i\in I \text{ and } f\in F.$$

Hence, for all $m\geq 1$,

$$\Pr_{f^{(\infty)}}\{X_n\notin K \text{ for } 1\leq n\leq m|X_0=i\}\leq (1-\rho)^{[m/\nu]} \text{ for all } i\in I \text{ and } f\in F,$$

using the fact that this probability is non-increasing in m. Next by the relation

$$(2.16)\qquad E_{f^{(\infty)}}\{N_K|X_0=i\}=1+\sum_{m=1}^{\infty} Pr_{f^{(\infty)}}\{X_n\notin K \text{ for } 1\leq n\leq m|X_0=i\},\ i\in I \text{ and } f\in F,$$

we get (ii).

Suppose next that (ii) holds. We shall now verify (iii). Fix $0<\varepsilon<1$ and choose $0<\gamma<1$ such that $(1-\gamma)^2\geq 1-\varepsilon$. Then we can find an integer $N\geq 1$ such that

$$(2.17)\qquad Pr_{f^{(\infty)}}\{X_n\notin K \text{ for } 1\leq n\leq N|X_0=i\}\leq\gamma \quad \text{for all } i\in I \text{ and } f\in F.$$

To prove this, suppose that for each integer $m\geq 1$ there exists a state $i\in I$ and a $f\in F$ such that $Pr_{f^{(\infty)}}\{X_n\notin K \text{ for } 1\leq n\leq m|X_0=i\}>\gamma$. Since this probability is non-increasing in m, it follows from (2.16) that $E_{f^{(\infty)}}\{N_K|X_0=i\}>1+m\gamma$ which contradicts (2.14). Hence (2.17) holds. We next show that there is a finite set A such that

$$(2.18)\qquad \sum_{j\in A} p_{ij}^m(f)\geq 1-\gamma \quad \text{for all } i\in K,\ 1\leq m\leq N \text{ and } f\in F.$$

To do this, fix $i\in K$ and $1\leq k\leq N$. In the same way as in the second part of the proof of Theorem 2.1, we find that for each $\gamma>0$ there is a finite set $A(\gamma)$ such that

$$\sum_{j\in A(\gamma)} p_{ij}^k(f)\geq 1-\gamma \quad \text{for all } f\in F.$$

Using this result and the finiteness of the set K, we obtain (2.18). Now, by (2.17) and (2.18) we find for all $i\in I$ and $f\in F$,

$$\sum_{j\in A} p_{ij}^{N+1}(f)\geq \sum_{n=1}^{N}\ \sum_{k\in K} Pr_{f^{(\infty)}}\{X_n=k, X_m\notin K \text{ for } 1\leq m\leq n-1|X_0=i\}\ \sum_{j\in A} p_{kj}^{N+1-n}(f)\ \geq$$

$$\geq (1-\gamma)Pr_{f^{(\infty)}}\{X_n\in K \text{ for some } 1\leq n\leq N|X_0=i\}\geq(1-\gamma)^2\geq 1-\varepsilon$$

which verifies (iii) since ε was arbitrarily chosen. Finally, it is immediate that (iii) implies (i).

Theorem 2.2 has the following corollary.

THEOREM 2.3 (cf. [4]). *Suppose that condition C3 holds without the requirement that for any* $f\in F$ *the stochastic matrix* $P(f)$ *has no two disjoint closed sets. Then for any* $\varepsilon>0$ *there is a finite set* $K(\varepsilon)$ *such that*

$$\sum_{j\in K(\varepsilon)} \pi_{ij}(f)\geq 1-\varepsilon \textit{ for all } i\in I \textit{ and } f\in F,$$

i.e. $[\{\pi_{ij}(f),j\in I\}|i\in I,f\in F]$ *is a tight collection of probability distributions.*

PROOF. Using Theorem 2.2 and relation (2.6), we have that for any $\varepsilon>0$ there is a finite set $K(\varepsilon)$ and an integer $\nu(\varepsilon)\geq 1$ such that

$$\sum_{j\in K(\varepsilon)} p^n_{ij}(f)\geq 1-\varepsilon \text{ for all } i\in I,\ f\in F \text{ and } n\geq\nu(\varepsilon).$$

Together this relation and (2.7) imply the Theorem.

The proof of the next theorem does not require assumption A1.

THEOREM 2.4 (cf. [1] and [3]). *Condition C4 implies condition C5.*

PROOF. Let C4 holds with pair (ν,ρ). Fix $f\in F$ and $A\subseteq I$. For $n=1,2,...,$ define

$$M_n=\sup_{i\in I}\sum_{j\in A} p^n_{ij}(f) \text{ and } m_n=\inf_{i\in I}\sum_{j\in A} p^n_{ij}(f).$$

Using (2.6), it follows that

$$(2.19)\qquad M_{n+1}\leq M_n \text{ and } m_{n+1}\geq m_n \quad \text{for all } n\geq 1.$$

For any number a, let $a^+=\max(a,0)$ and $a^-=-\min(a,0)$. Then $a^+,a^-\geq 0$ and $a=a^+-a^-$. For any sequence $\{a_j,j\in I\}$ of numbers such that $\Sigma_{j\in I}|a_j|<\infty$ and $\Sigma_{j\in I}a_j=0$, we have $\Sigma_{j\in I}a_j^+ = \Sigma_{j\in I}a_j^-$. Further, we note that $(a-b)^+ = a-\min(a,b)$ for any numbers a,b. Fix now $i\in I$ and $n>\nu$. Then

$$\sum_{j\in A} p_{ij}^{n}(f) - \sum_{j\in A} p_{rj}^{n}(f) = \sum_{k\in I} \{p_{ik}^{\nu}(f) - p_{rk}^{\nu}(f)\} \sum_{j\in A} p_{kj}^{n-\nu}(f) =$$

$$= \sum_{k\in I} \{p_{ik}^{\nu}(f) - p_{rk}^{\nu}(f)\}^{+} \sum_{j\in A} p_{kj}^{n-\nu}(f) - \sum_{k\in I} \{p_{ik}^{\nu}(f) - p_{rk}^{\nu}(f)\}^{-} \sum_{j\in A} p_{kj}^{n-\nu}(f) \leq$$

$$\leq \{M_{n-\nu} - m_{n-\nu}\} \sum_{k\in I} \{p_{ik}^{\nu}(f) - p_{rk}^{\nu}(f)\}^{+} =$$

$$= \{M_{n-\nu} - m_{n-\nu}\}\{1 - \sum_{k\in I} \min[p_{ik}^{\nu}(f), p_{rk}^{\nu}(f)]\} \leq$$

$$\leq (1-\rho)(M_{n-\nu} - m_{n-\nu}).$$

Since i and r were arbitrarily chosen, it follows that

$$M_n - m_n \leq (1-\rho)\{M_{n-\nu} - m_{n-\nu}\} \quad \text{for all } n>\nu.$$

Hence, since $M_n - m_n$ is non-increasing in $n\geq 1$,

$$\text{(2.20)} \qquad M_n - m_n \leq (1-\rho)^{[n/\nu]} \quad \text{for all } n\geq 1.$$

Together (2.19) and (2.20) imply that for some finite non-negative number $\pi(A)$

$$\lim_{n\to\infty} M_n = \lim_{n\to\infty} m_n = \pi(A).$$

Further for any $n\geq 1$,

$$\text{(2.21)} \qquad m_n \leq \pi(A) \leq M_n \quad \text{and} \quad m_n \leq \sum_{j\in A} p_{ij}^{n}(f) \leq M_n \quad \text{for all } i\in I.$$

It now follows from (2.20) and (2.21) that

$$\Big| \sum_{j\in A} p_{ij}^{n}(f) - \pi(A) \Big| \leq (1-\rho)^{[n/\nu]} \quad \text{for all } n\geq 1.$$

Since this relation holds for any $A\subseteq I$, it follows that $\pi\{.\}$ is a probability measure on the class of all subsets of I which completes the proof.

THEOREM 2.5 (cf. [3]) *The condition C3 together with the assumption that for each* $f\in F$ *the stochastic matrix* $P(f)$ *is aperiodic is equivalent to each of the conditions C4 and C5.*

PROOF. Suppose first that C3 with triple (K,ν,ρ) holds and that any $P(f)$ is aperiodic. We shall then verify condition C4. Since for any $f\in F$ the stochastic matrix $P(f)$ satisfies the Doeblin condition, has no two disjoint closed sets and is aperiodic, we have from Markov chain theory (e.g. [2]) that

$$(2.22)\qquad \lim_{n\to\infty} p^n_{ij}(f) = \pi_j(f) \quad \text{for all } i,j\in I.$$

Since (2.3) implies $\Sigma_{j\in K} p^n_{ij}(f)\geq\rho$ for all $i\in I$, $f\in F$ and $n\geq\nu$, we have

$$(2.23)\qquad \sum_{j\in K}\pi_j(f)\geq\rho \quad \text{for all } f\in F.$$

Define now

$$(2.24)\qquad F_k = \{f\in F \mid \pi_k(f) \geq \frac{\rho}{|K|}\} \quad \text{for } k\in K,$$

where $|K|$ denotes the number of states in K. Then, by (2.23),

$$F = \bigcup_{k\in K} F_k.$$

Using the Theorems 2.1 and 2.3 and the fact that F is a compact metric space, it follows that for any $k\in K$ the set F_k is closed and hence compact. For any $i\in I$ and $k\in K$, define

$$(2.25)\qquad n(i,k,f) = \min\{n\geq 1 \mid p^n_{ik}(f) > \frac{\rho}{2|K|}\} \quad \text{for } f\in F_k.$$

By (2.22), $n(i,k,f)$ exists and is finite. Using the fact that $P^n(f)$ is continuous on F for each $n\geq 1$, it is immediately verified that for each $i\in I$ and $k\in K$ the set $\{f\in F_k \mid n(i,k,f)\geq\alpha\}$ is closed for any real α, i.e. for each $i\in I$ and $k\in K$ the function $n(i,k,f)$ is upper semi-continuous on the compact set F_k.

Now, by Proposition 10 on p. 161 in [11], we have that for each $i\in I$ and $k\in K$ the function $n(i,k,f)$ assumes a finite maximum on F_k. Hence, using the finiteness of K, we can find an integer $\mu\geq 1$ such that

(2.26) $\quad n(i,k,f)\leq\mu$ for all $i\in K$, $k\in K$ and $f\in F_k$.

Next define for any $k\in K$

(2.27) $\quad m(k,f) = \min\{n\geq 1 \mid p_{kk}^{m}(f) > \frac{\rho}{2|K|}$ for all $n\leq m\leq n+\mu\}$ $\quad$ for $f\in F_k$.

We now verify that for each $k\in K$ the set $S_\alpha=\{f\in F_k \mid m(k,f)\geq\alpha\}$ is closed for any real α. Fix $k\in K$ and an integer $\alpha>1$. Suppose that $f_n\in S_\alpha$ for $n\geq 1$ and that $f_n\to f^*$ as $n\to\infty$. Then we can find a subsequence $\{n_h, h\geq 1\}$ of integers and integers r and t with $1\leq r\leq\alpha-1$ and $r\leq t\leq r+\mu$ such that $p_{kk}^{t}(f_{n_h})\leq\rho/2|K|$ for all $h\geq 1$. Hence, by the fact that $p_{kk}^{t}(f)$ is continuous on F, we find $p_{kk}^{t}(f^*)\leq\rho/2|K|$ and so $f^*\in S_\alpha$. We have now proved that for any $k\in K$ the function $m(k,f)$ is upper semi-continuous on the compact set F_k. Hence there exists an integer $N\geq 1$ such that

$$m(k,f)<N \quad \text{for all } k\in K \text{ and } f\in F_k.$$

For any $k\in K$ and $f\in F_k$, we have by (2.25)-(2.27)

$$p_{ik}^{\mu+m(k,f)}(f)\geq p_{ik}^{n(i,k,f)}(f)\,p_{kk}^{m(k,f)+\mu-n(i,k,f)}(f) > \frac{\rho^2}{4|K|^2} \quad \text{for all } i\in K.$$

Hence, for any $k\in K$ and $f\in F_k$,

$$p_{ik}^{\nu+\mu+m(k,f)}(f) \geq \sum_{j\in K} p_{ij}^{\nu}(f)\,p_{jk}^{\mu+m(k,f)}(f) > \frac{\rho^3}{4|K|^2} \quad \text{for all } i\in I.$$

Using this result, we now find for any $k\in K$ and $f\in F_k$,

$$\sum_{j\in I}\min[p_{i_1 j}^{\nu+\mu+N}(f),\ p_{i_2 j}^{\nu+\mu+N}(f)] \geq$$

$$\geq \sum_{j\in I}\min[p_{i_1 k}^{\nu+\mu+m(k,f)}(f)\,p_{kj}^{N-m(k,f)}(f),\ p_{i_2 k}^{\nu+\mu+m(k,f)}(f)\ p_{kj}^{N-m(k,f)}(f)] \geq$$

$$\geq \frac{\rho^3}{4|K|^2} \sum_{j\in I} p_{kj}^{N-m(k,f)}(f) = \frac{\rho^3}{4|K|^2} \quad \text{for all } i_1,i_2\in I,$$

which verifies C4.

By Theorem 2.4 we have that condition C4 implies condition C5. Suppose now that condition C5 holds. Then any P(f), $f\in F$ is aperiodic. To complete the proof, we now verify that condition C3 holds. Since $P^n(f)$ is continuous on F for each $n\geq 1$, it follows from (2.5) that for any $j\in I$ the function $\pi_j(f)$ is continuous on F. By Theorem 2.1, we now have that any $\varepsilon>0$ there is a finite set $K(\varepsilon)$ such that (2.10) holds. Next by using the uniform convergence in (2.5), we find that for any $\varepsilon>0$ there is a finite set $K(\varepsilon)$ and an integer $\nu(\varepsilon)\geq 1$ such that (2.15) holds. Now, by Theorem 2.2, we find that condition C3 holds which completes the proof.

THEOREM 2.6 *The conditions C1, C2 and C3 are equivalent.*

PROOF. By Theorem 2.2, C1 and C3 are equivalent. Suppose now that C3 holds with triple (K,ν,ρ). We shall verify C2. As in the first part of the proof of Theorem 2.5, we again obtain relation (2.23) and the compactness of the set F_k for any $k\in K$ where F_k is defined by (2.24). Fix now $k\in K$. For any $f\in F_k$, define the stochastic matrix $\hat{P}(f)=(\hat{p}_{ij}(f))$, $i,j\in I$ by

(2.28) $\hat{p}_{ij}(f) = p_{ij}(f)$ for $i\neq k$, $j\in I$ and $\hat{p}_{kk}(f)=1$.

Denote by $\hat{P}^n(f)$ the n-fold matrix product of $\hat{P}(f)$ with itself for $n\geq 1$. Using the fact that P(f) is continuous on F, it is immediately verified by induction that $\hat{P}^n(f)$ is continuous on F_k for each $n\geq 1$. By the definition (2.28), we have for any $f\in F_k$ that the expected number of transitions until the first return to state k under $\hat{P}(f)$ is equal to that under P(f) for any initial state $i\neq k$. Hence, by the finiteness of K and the fact that $\cup_{k\in K} F_k = F$, it suffices to prove that there is a finite number B_k such that for each $f\in F_k$ the expected number of transitions until the first return to state k under $\hat{P}(f)$ is less than or equal to B_k for each initial state $i\in I$. To prove this, we first observe that, by (2.28) and the fact that $k\in K$, we have

(2.29) $\sum_{j\in K} \hat{p}_{ij}^{\nu}(f) \geq \sum_{j\in K} p_{ij}^{\nu}(f) \geq \rho$ for all $i\in I$ and $f\in F_k$,

i.e. $\hat{P}(f)$ satisfies the Doeblin condition.

Since for any $f\in F_k$ we have that under $P(f)$ state k is positive recurrent and hence can be reached from any other state, it follows for any $f\in F_k$ that the stochastic matrix $\hat{P}(f)$ has no two disjoint closed sets and that under $\hat{P}(f)$ any state $i\neq k$ is transient and state k is an aperiodic positive recurrent state. Since $\hat{P}(f)$ also satisfies the Doeblin condition, we have from Markov chain theory (e.g. [2]) that for any $f\in F_k$

$$\lim_{n\to\infty} \hat{p}^n_{ik}(f) = 1 \quad \text{for all } i\in I.$$

Define now for any $i\in I$

$$n(i,f) = \min\{n\geq 1 \mid \hat{p}^n_{ik}(f) > \tfrac{1}{2}\} \quad \text{for } f\in F_k$$

Since $\hat{P}^n(f)$ is continuous on F_k, it follows for any $i\in I$ the finite function $n(i,f)$ is upper semi-continuous on the compact set F_k. Hence there is an integer $\mu_k\geq 1$ such that

(2.30) $\quad n(i,f)\leq\mu_k \quad$ for all $i\in K$ and $f\in F_k$.

We shall now verify that

(2.31) $\quad \hat{p}^{\nu+\mu_k}_{ik}(f) > \dfrac{\rho}{2|K|}$ for all $i\in I$ and $f\in F_k$.

To do this, observe that, by (2.29), for any $i\in I$ and $f\in F_k$ we can find a state $j\in K$ such that $\hat{p}^{\nu}_{ij}(f) \geq \rho/|K|$ and so $\hat{p}^{\nu+n(j,f)}_{ik} \geq \hat{p}^{\nu}_{ij}(f)\hat{p}^{n(j,f)}_{jk} > \rho/2|K|$. This relation and (2.30) imply (2.31) since state k is absorbing under $\hat{P}(f)$. From (2.31) it follows for any $f\in F_k$ that the expected number of transitions until the first return to state k under $\hat{P}(f)$ is less than or equal to $2|K|(\nu+\mu_k)/\rho$ for any starting state $i\in I$ which completes the proof.

Finally we show that in condition C1 the set K can be taken as a singleton when the stochastic matrices $P(f)$, $f\in F$ have a common recurrent state.

THEOREM 2.7. (cf.[4]). (a) *Suppose that condition C3 holds without the requirement that any* $P(f)$, $f\in F$ *has no two disjoint closed sets. Let* $A\subseteq I$ *and the compact set* $G\subseteq F$ *be such that for each* $i\in I$ *and* $f\in G$ *there exists a state* $j\in A$ *and an integer* $n\geq 1$ *for which* $p^n_{ij}(f)>0$. *Then there is a finite number* B *such that*

$E_{f^{(\infty)}}\{N_A|X_0=i\}\le B$ *for all* $i\in I$ *and* $f\in G$.

(b) *Suppose that there is a state* $i_0\in I$ *such that for any* $i\in I$ *and* $f\in F$ *there exists an integer* $n\ge 1$ *for which* $p^n_{ii_0}(f)>0$. *Then in condition C1 the set K can be taken equal to the singleton* $\{i_0\}$.

PROOF. (a) Let (K,ν,ρ) be the triple in C3. For each $i\in I$, define

$$n(i,f)=\min\{n\ge 1|\sum_{j\in A}p^n_{ij}(f)>0\}\quad \text{for } f\in G.$$

It is readily verified that for each $i\in I$ the finite function $n(i,f)$ is upper semi-continuous on the compact set G. Hence we can find an integer $\mu\ge 1$ such that $n(i,f)\le\mu$ for all $i\in K$ and $f\in G$, so

$$\Pr_{f^{(\infty)}}\{X_n\in A \text{ for some } 1\le n\le\mu|X_0=i\}>0\quad \text{for all } i\in K \text{ and } f\in G.$$

Since for each $i\in K$ this probability is a continuous function in $f\in G$ and G is compact, there exists a number $\alpha>0$ such that

$$\Pr_{f^{(\infty)}}\{X_n\in A \text{ for some } 1\le n\le\mu|X_0=i\}\ge\alpha\quad \text{for all } i\in K \text{ and } f\in G.$$

We now find

$$\Pr_{f^{(\infty)}}\{X_n\in A \text{ for some } 1\le n\le\nu+\mu|X_0=i\}\ge$$

$$\ge \sum_{j\in K}p^{\nu}_{ij}(f)\Pr_{f^{(\infty)}}\{X_n\in A \text{ for some } 1\le n\le\mu|X_0=j\}\ge\alpha\rho\quad \text{for all } i\in I \text{ and } f\in G.$$

Hence $\Pr_{f^{(\infty)}}\{X_n\notin A \text{ for } 1\le n\le\nu+\mu|X_0=i\}\le 1-\alpha\rho$ for all $i\in I$ and $f\in G$ which implies part (a) of the Theorem with $B=(\nu+\mu)/\alpha\rho$.

(b) This part is an immediate consequence of Theorem 2.2 and part (a) of Theorem 2.7.

REMARK. In Theorem 2.6 it was proved that C3 implies C2. Alternatively, this result may be obtained by considering the compact sets F_k defined in (2.24) which have the property that state k can be reached from any other state under P(f) for $f\in F_k$ and by applying part (a) of Theorem 2.7.

3. THE OPTIMALITY EQUATION.

In this section we shall discuss the optimality equation for the average costs. As a consequence of the Theorems 2.5 and 2.6 we have that each of the conditions C1-C5 implies condition C2. In the next theorem we shall prove under a slight weakening of condition C2 that the optimality equation for the average costs has a bounded solution.

THEOREM 3.1 (cf. [3], [4] and [10]). *Suppose that a finite number* B *exists such that for any* $f \in F$ *there is a state* s_f *for which*

$$E_{f^{(\infty)}}\{N_{\{s_f\}} | X_0 = i\} \le B \text{ for all } i \in I.$$

Then there exists a constant g *and a bounded function* v(i), $i \in I$ *such that*

$$(3.1) \qquad v(i) = \min_{a \in A(i)} \{c(i,a) - g\tau(i,a) + \sum_{j \in I} p_{ij}(a) v(j)\} \text{ for all } i \in I$$

PROOF. To establish (3.1) it is no restriction to assume that the times between the decision epochs are deterministic, since in (3.1) the transition times only appear through their expectations. Now, we first consider the discounted cost model. For any $\alpha > 0$, define for each policy π

$$V_\alpha(i,\pi) = E_\pi\{\sum_{n=0}^{\infty} e^{-\alpha(\tau_0 + \ldots + \tau_n)} c(X_n, a_n) | X_0 = i\} \quad \text{for } i \in I,$$

where $\tau_0 = 0$ and, for $n \ge 1$, τ_n denotes the time between the (n-1)st and nth decision. Further, for any $\alpha > 0$, let $V_\alpha(i) = \inf_\pi V_\alpha(i,\pi)$ for $i \in I$. The above quantities are well-defined. Letting the constants M and δ be as in the assumptions A2 and A3, we have for any $\alpha > 0$ and policy π that $|V_\alpha(i,\pi)| \le M/(1-e^{-\alpha\delta})$ for all $i \in I$. Hence, since $\alpha/(1-e^{-\alpha\delta}) \to 1/\delta$ as $\alpha \to 0$, we can find a number $\alpha^* > 0$ such that

$$(3.2) \qquad |\alpha V_\alpha(i,\pi)| \le \frac{2M}{\delta} \quad \text{for any } i \in I,\ 0 < \alpha < \alpha^* \text{ and policy } \pi.$$

Using known results for the discounted cost model (see [4], [8] and [13]), we have that for any $\alpha > 0$ the function $V_\alpha(i)$, $i \in I$ is the unique bounded solution to

$$(3.3) \qquad V_\alpha(i) = \min_{a \in A(i)} \{c(i,a) + e^{-\alpha\tau(i,a)} \sum_{j \in I} p_{ij}(a) V_\alpha(j)\} \quad \text{for } i \in I.$$

Moreover, for any $\alpha>0$, there exist a $f_\alpha \in F$ such that

$$(3.4) \qquad V_\alpha(i,f_\alpha^{(\infty)}) = V_\alpha(i) \quad \text{for all } i\in I$$

and $f_\alpha \in F$ satisfies (3.4) if and only if $f_\alpha(i)$ minimizes the right side of (3.3) for all $i\in I$. We shall now verify that there is a finite number γ such that

$$(3.5) \qquad |V_\alpha(i)-V_\alpha(j)|\le\gamma \quad \text{for all } i,j\in I \text{ and } 0<\alpha<\alpha^*.$$

To do this, choose $0<\alpha<\alpha^*$ and $f\in F$. Then, letting $N = N_{\{s_f\}}$,

$$V_\alpha(i,f^{(\infty)})-V_\alpha(s_f,f^{(\infty)})=E_{f^{(\infty)}}\{\sum_{n=0}^{N-1} e^{-\alpha(\tau_0+\ldots+\tau_n)} c(X_n,a_n)|X_0=i\} +$$
$$+ V_\alpha(s_f,f^{(\infty)})E_{f^{(\infty)}}\{e^{-\alpha(\tau_0+\ldots+\tau_N)}|X_0=i\}-V_\alpha(s_f,f^{(\infty)}), \; i\in I.$$

Next, using the fact that $1-e^{-x}\le x$ for $x\ge0$ and (3.2), we obtain

$$|V_\alpha(i,f^{(\infty)})-V_\alpha(s_f,f^{(\infty)})|\le MB+MB|\alpha V_\alpha(s_f,f^{(\infty)})| \le$$
$$\le MB + \frac{2M^2B}{\delta} \quad \text{for all } i\in I,$$

Together, this relation and (3.4) imply (3.5) since α and f were arbitrarily chosen. Fix now any state $r\in I$ and define for any $\alpha>0$

$$h_\alpha(i) = V_\alpha(i)-V_\alpha(r) \quad \text{for } i\in I.$$

Then (3.3) can be equivalently written as

$$(3.6) \qquad h_\alpha(i)=\min_{a\in A(i)} \{c(i,a)+e^{-\alpha\tau(i,a)} \sum_{j\in I} p_{ij}(a)h_\alpha(j) + \frac{1}{\alpha} (e^{-\alpha\tau(i,a)}-1)\alpha V_\alpha(r)\}, \; i\in I.$$

For any $\alpha>0$, let $f_\alpha \in F$ be such that $f_\alpha(i)$ minimizes the right side of (3.6) for all $i\in I$. Now, observe that by (3.2) and (3.5), both $h_\alpha(i)$ and $\alpha V_\alpha(i)$ are uniformly bounded in $i\in I$ and $0<\alpha<\alpha^*$. Using the well-known diagonalization method and the fact that $A(i)$ is a compact metric space for any $i\in I$, we can find a sequence $\{\alpha_n,n\ge1\}$ of numbers with $\alpha_n\to0$ as $n\to\infty$, a function $f^*\in F$ and a finite constant g and a bounded function $v(i)$, $i\in I$ such that

$$\lim_{n\to\infty} \alpha_n V_{\alpha_n}(r)=g,\ \lim_{n\to\infty} h_{\alpha_n}(i)=v(i) \text{ and } \lim_{n\to\infty} f_{\alpha_n}(i)=f^*(i) \quad \text{for all } i\in I.$$

Now, for any $n\geq 1$ and $i\in I$, we have

$$h_{\alpha_n}(i)\leq c(i,a)+e^{-\alpha_n \tau(i,a)} \sum_{j\in I} p_{ij}(a)h_{\alpha_n}(j) + \frac{1}{\alpha_n}(1-e^{-\alpha_n \tau(i,a)})\alpha_n V_{\alpha_n}(r)$$

$$\text{for } a\in A(i),$$

where the equality sign holds for $a=f_{\alpha_n}(i)$. Now, letting $n\to\infty$, using assumption A1 and Proposition 18 on p. 232 in [11], we find for any $i\in I$

$$v(i)\leq c(i,a)+ \sum_{j\in I} p_{ij}(a)v(j)-\tau(i,a)g \quad \text{for } a\in A(i)$$

where the equality sign holds for $a=f^*(i)$. This gives (3.1).

We end this paper by making some remarks. We first remark that, by using a data transformation introduced in [14] and results in [5], it was shown in [3] that value iteration may be used to determine a bounded solution to the optimality equation (3.1) under each of the conditions C1-C5. Further, it was proved in [3] that under condition C1 with K a singleton the policy iteration algorithm generates a sequence of stationary policies for which both the associated average costs and relative cost functions converge so that the limits satisfy the optimality equation.

We next remark that a repeated application of the result of Theorem 3.1 gives a sequence of optimality equations that are involved when considering the more sensitive and selective n-discounted optimality criteria, cf. [6] and [7].

Finally we remark that so far we have assumed that both $c(i,a)$ and $\tau(i,a)$ are uniformly bounded in i,a. For the case in which only the assumptions A1 and A3 are made, it was shown in chapter 5 of [4] that the optimality equation (3.1) has a finite solution under the following condition C6.

C6. *There exists a state* s *and finite non-negative numbers* y_i, $i\in I$ *such that*
(a) $|c(i,a)|+\tau(i,a)+\Sigma_{j\in I}\hat{p}_{ij}(a)y_j\leq y_i$ *for all* $i\in I$ *and* $a\in A(i)$,
(b) *For any* $i\in I$, $\Sigma_{j\in I}\hat{p}_{ij}(a)y_j$ *is continuous on* $A(i)$,
(c) $\lim_{n\to\infty} \Sigma_{j\in I}\hat{p}^n_{ij}(f)y_j=0$ *for all* $i\in I$ *and* $f\in F$,

where, for all $i,j \in I$ *and* $a \in A(i)$,

$$\hat{p}_{ij}(a)=p_{ij}(a) \text{ if } i \neq s \text{ and } \hat{p}_{ij}(a)=0 \text{ if } i=s,$$

and, for any $f \in F$, $\hat{p}^{n}_{ij}(f)$ *is the n-fold matrix product of the matrix* $(\hat{p}_{ij}(f(i)))$, $i,j \in I$ *with itself*

It was shown in chapter 12 of [4] that in case assumption A2 does holds the condition C6 with a bounded function y_i, $i \in I$ is equivalent to the condition C1 with the set K consisting of a single state. The Liapunov function approach given by condition C6 was further investigated in [6] and [7] where in particular sensitive optimality criteria were studied.

REFERENCES

1. ANTHONISSE, J.M. and TIJMS, H.C. (1977), Exponential convergence of products of stochastic matrices, *J. Math. Anal. Appl.* 59, 360-364.
2. DOOB, J.L. (1953), *Stochastic Processes*, Wiley, New York.
3. FEDERGRUEN, A. and TIJMS, H.C. (1977), The optimality equation in average cost denumerable state semi-Markov decisions problems, recurrency conditions and algorithms (to appear in *J. Appl. Prob.*).
4. HORDIJK, A. (1974), *Dynamic Programming and Markov Potential Theory*, Mathematical Centre Tract no. 51, Mathematisch Centrum, Amsterdam.
5. HORDIJK, A. SCHWEITZER, P.J. and TIJMS, H.C. (1975), The asymptotic behaviour of the minimal total expected cost for the denumberable state Markov decision model, *J. Appl. Prob.* 12, 298-305.
6. HORDIJK, A. and SLADKY, K. (1975), Sensitive optimality criteria in countable state dynamic programming (to appear in *Math. of Operations Res.*).
7. HORDIJK, A. (1976), Regenerative Markov decision models, *Mathematical Programming Study* 6, 49-72, North-Holland, Amsterdam.
8. MAITRA, A. (1968), Discounted dynamic programming on compact metric spaces, *Sankhya Ser. A.* 30, 211-216.
9. NEVEU, J. (1965), *Mathematical Foundations of the Calculus of Probability*, Holden-Day, San Francisco.
10. ROSS, S.M. (1970), *Applied Probability Models with Optimization Applications*, Holden-Day, San Francisco.
11. ROYDEN, H.L. (1968), *Real Analysis* (2nd. edition), MacMillan, New York.
12. RUDIN, W. (1964), *Principles of Mathematical Analysis* (2nd. ed.), McGraw Hill, New York.
13. SCHÄL, M. (1975), Conditions for optimality in dynamic programming and for the limit of n-stage optimal policies to be optimal, *Z. Wahrscheinlichkeitsth.* 32, 179-196.
14. SCHWEITZER, P.J. (1971), Iterative solution of the functional equations of undiscounted Markov renewal programming, *J. Math. Anal. Appl.* 34, 495-501.
15. SENETA, E. (1974), *Non-negative Matrices*, George Allen & Unwin Ltd. London.